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NASA Contractor Report 181734

ICASE REPORT NO. 88-62

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A PHYSICALLY MOTIVATED DOMAIN DECOMPOSITION FOR SINGULARLY PERTURBED EQUATIONS

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Contract No. NAS1-18605
October 1988

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Operated by the Universities Space Research Association

National Aeronautics and
Space Administration

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A PHYSICALLY MOTIVATED DOMAIN DECOMPOSITIONS FOR SINGULARLY PERTURBED EQUATIONS

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ABSTRACT

A domain decomposition algorithm suitable for the efficient and accurate solution of a parabolic reaction convection diffusion equation with small parameter on the diffusion term is presented. Convergence is established via maximum principle arguments. The equation arises in the modeling of laminar transonic flow. Decomposition into subdmomains is accomplished via singular perturbation analysis which dictates regions where certain reduced equations may be solved in place of the full equation, effectively preconditioning the problem. This paper concentrates on the the theoretical basis of the method, establishing local and global *a priori* error bounds.

[1] Research conducted while in residence at the Center for Supercomputing Research and Development, University of Illinois, supported in part by the National Science Foundation under Grant No. US NSF PIP-8410110, the U.S Department of Energy under Grant No. US DOE-DE-FG02-85ER25001, the Air Force Office of Scientific Research under Grant No. AFOSR-85-0211, the IBM Donation to CSRD, and by the Applied Mathematical Sciences subprogram of the Office of Energy Research, U.S. Department of Energy by Lawrence Livermore National Laboratory under contract No. W-7405-Eng-48. Research was also partially supported by NASA Contract No. NAS1-18605 while in residence at ICASE.

1. Introduction. In this paper a domain decomposition algorithm for the solution of

$$P[u] := u_t + uu_x - \epsilon u_{xx} - ru = 0, \tag{1}$$

where ϵ is a small positive parameter is presented. This equation contains many of the properties that make the gasdynamic equations difficult to solve; namely, it is capable of modeling rapid variations such as shocks and boundary layers. *A priori* error bounds are obtained using asymptotic analysis, and are verified via maximum principle arguments. The analysis also identifies parallelism intrinsic in the physics of the problem. This parallelism may be exploited by the particular numerical methods, allowing efficient use of parallel architectures. This paper concentrates on the theoretical basis for the method, discussions of the numerical details may be found in [10] and [15].

The method presented here is appropriate for certain problems arising when modeling laminar transonic flow, such as through a duct of variable width. When modeling transonic flow, except in regions of rapid variation such as in shocks and boundary layers, convection and/or reaction terms dominate over diffusion. The reaction term may, for example, arise from the effects of a variable cross sectional area in a duct, thus this not a reacting flow. Asymptotic analysis identifies the regions where the solution behaves different, subdividing the domain into the following two types of regions: regions where the solution is smooth, where a reduced equation may be solved; and regions of rapid variations, such as in a neighborhood of a shock, where the full equation must be solved. The domain decomposition is independent of the choice of numerical schemes for the subdomains, hence the numerics will be discussed in this paper only briefly. In addition to dictating the domain decomposition, asymptotics also provides a means of approximating solutions to the problems in the subdomains. In this way, a set of simplified problems is obtained that is better conditioned for numerical computations. The domain decomposition and preconditionings are reflected in the theorems presented herein.

The asymptotic analysis involves the derivation of analytic upper and lower bounds on the solution, and is performed in the style of Howes [4,5,6]. The method is capable of obtaining solutions to (1) when the shock is not stationary, thus extending Howes' studies [7,8] into the time-dependent regime.

The method is an iterative technique. In Section 3 the domain decomposition and some preconditionings are presented. This includes an error analysis of the preconditionings and the theoretical basis of the domain decomposition. In Section 5, the method is summarized by outlining the algorithm.

2. The Quasilinear Problem. Consider the behavior of the solution of the quasilinear parabolic equation (1) on the domain

$$D := \{(x,t) | 0 \le x \le b, 0 \le t < T\}, \tag{2}$$

subject to

$$u(x,0) = \gamma(x), \quad 0 < x < b; \tag{3}$$

$$u(0,t) = \alpha(t), \quad 0 < t < T; \text{ and} \tag{4}$$

$$u(b,t) = \beta(t), \quad 0 < t < T. \tag{5}$$

The portion of the boundary along which the data is specified is denoted by

$$\Pi := \{(x,t)|0 \le x \le b,\ t = 0\} \bigcap \{(x,t)|0 \le t < T,\ x = 0, b\}.$$

For the sake of simplicity, it is assumed that all boundaries are inflow boundaries, that is, $\alpha(t) \ge \alpha_0 > 0$ and $\beta(t) \le \beta_0 < 0$.

The reaction term may, for example, arise from the effects of a variable cross sectional area in a duct. Howes [8] discusses the case when $r(x) = -a'(x)/a(x)$, where $a(x)$ is the width of the duct (see Figure 1). The coefficient r is assumed to be bounded

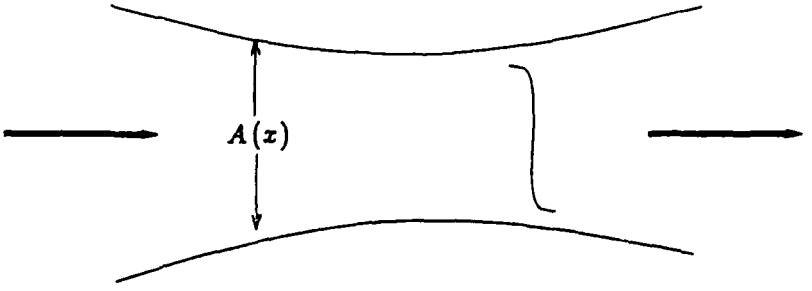

FIG. 1. *Variable width duct.*

with bounded derivatives.

It is assumed that the boundary data are sufficiently smooth so that the solution to (1) is uniquely defined (for example, see [2]). We are interested in the formation of shocks, thus the data is assumed to be continuous. For example, the compatibility conditions

$$\alpha(0) = \gamma(0), \quad \text{and} \quad \gamma(b) = \beta(0), \tag{6}$$

must be satisfied. In addition, it is assumed that the first derivatives of the solution to the reduced ($\epsilon = 0$) equation

$$P_0[U] := U_t + UU_x - rU = 0 \tag{7}$$

are continuous except along the shock. This requires, for example,

$$\frac{d\alpha}{dt} + \gamma\frac{d\gamma}{dx} - r\gamma = 0, \quad \text{for } (x,t) = (0,0); \tag{8}$$

$$\frac{d\beta}{dt} + \gamma\frac{d\gamma}{dx} - r\gamma = 0, \quad \text{for } (x,t) = (b,0). \tag{9}$$

The problem is assumed to be nondimensionalized such that the diffusion coefficient ϵ is inversely proportional to the Reynolds number (see [13]). Based on free-stream conditions in transonic flow, the Reynolds number for this problem is large. Therefore, it is appropriate to exploit the smallness of the positive parameter ϵ in the analysis.

3. Analysis of the Preconditionings. The method exploits preconditionings to obtain computational efficiency and accuracy. The meaning of preconditioning is broader than the usual meaning applied in the linear algebra setting. In this setting, a problem is preconditioned if it is more tractable with respect to numerical computations. For example, in this section a preconditioner based on a physically motivated domain decomposition is discussed. Regions where the solution behaves differently are identified. The problem is better conditioned because each numerical method used now needs only capture one type of behavior in the solution. In addition to the domain decomposition, a preconditioner involving a transformation of the spatial coordinate and a preconditioner involving a modification of the governing equation will be presented.

The domain decomposition and the use of the reduced equation are closely related. Asymptotic analysis identifies two types of regions. In the outer regions, the solution is slowly varying and the ϵu_{xx} term is small. Thus, in the outer region subdomains, the governing equation is modified by dropping this term with minor effects on the error. The solution to the reduced equation will be described next.

Let U be a weak solution of (7) with boundary data (3-5), which is a solution to (1) in the limit as $\epsilon \downarrow 0$. For the analysis here, assume that U has a single shock. Let the path of the shock be given by the curve $(x,t) = (\Gamma(t),t)$. The initial and boundary data are assumed to be smooth; thus, the shock does not exist at $t = 0$. Rather, Γ is assumed to be undefined for $t < t^\Gamma$, where $t = t^\Gamma$ is the time U becomes discontinuous. It is natural to describe U in terms of the following functions:

$$U(x,t) = \begin{cases} U_0(x,t) & \text{for } 0 < t \le t^\Gamma \\ U_l(x,t) & \text{for } x < \Gamma^{-1}(t) \text{ and } t \ge t^\Gamma \\ U_r(x,t) & \text{for } x > \Gamma^{-1}(t) \text{ and } t \ge t^\Gamma. \end{cases}$$

For analytic methods to choose Γ, see Whitham [19] or Kevorkian and Cole [9].

The shocks in the system are assumed to be physical; thus, the solution will satisfy the entropy condition

$$U_l(\Gamma(t),t) > s > U_r(\Gamma(t),t), \tag{10}$$

where the speed s of the shock is given by the Rankine-Hugoniot jump condition [11],

$$s = [U_l(\Gamma(t),t) + U_r(\Gamma(t),t)]/2. \tag{11}$$

The entropy condition may be written as

$$\mu(t) = U_l(\Gamma(t), t) - U_r(\Gamma(t), t) > 0 \tag{12}$$

for $t > t^{\Gamma}$.

The outer-region subdomains are dictated by the regions where U is a good approximation to u. These are defined by bounding the difference $U - u$. The bounds are reflected in the following theorem.

THEOREM 1. (Howes [6]). *Let $u(x,t,\epsilon)$ be the solution to $P[u] = 0$ on D and $U(x,t)$ be the solution to $P_0[U] = 0$ in the limit as ϵ tends to zero, each satisfying the the boundary data* (3-5). *Assume that the boundary data* (3-5) *satisfy the compatibility conditions* (6),(8-9), *and that α, β and γ, with their first and second derivatives are all bounded. Then for ϵ small enough*

$$|u - U| = O(\mu \exp[-f^2(x,t)/\epsilon^{1/2}]) + O(\epsilon) \tag{13}$$

when the derivatives of U are continuous across Γ, and

$$|u - U| = O(\mu \exp[-f^2/\epsilon^{1/2}]) + O(\epsilon^{1/2}\delta \exp[-f/\epsilon^{1/2}]) + O(\epsilon) \tag{14}$$

in the more general case when the derivatives of U are not continuous across Γ. Here $f(x,t)$ is a distance function between (x,t) and (Γ,t), and δ is an upper bound on the difference of the normal derivative of U across Γ.

The subdomains are dictated by the error bounds of this theorem. These bounds are small except in an asymptotically small neighborhood of the shock. The outer region subdomain is the portion of D where where using U to approximate u introduces a small error. The subdomain where U may be a poor approximation to u includes the internal or shock layer. The internal-layer subdomain is the following neighborhood of Γ:

$$D_{IL} = \{(x,t) | (x,t) \in D, |x - \Gamma^{-1}(t)| \leq \Delta(t)\}. \tag{15}$$

Here $\Delta(t)$ is the width of the internal-layer subdomain at time t. Theorem 1 dictates that $|u-U| = O(\nu)$ in D_{IL} when the internal-layer subdomain has size $O(\eta(t)\epsilon^{1/4}\ln^{1/2}\nu)$. Thus, to obtain an *a priori* bound of $O(\epsilon)$ on the error, the internal layer will be no larger than $\Delta(t) \leq K\eta(t)\epsilon^{1/4}\ln^{1/2}\epsilon$, where K is a constant independent of ϵ. The outer-region subdomain is the complement of D_{IL} with respect to D,

$$D_{OR} = \{(x,t) | (x,t) \in D, |x - \Gamma^{-1}(t)| > \Delta(t)\}. \tag{16}$$

Since the method is designed for small ϵ, the internal-layer subdomain will be an asymptotically small region surrounding the shock.

Since the solution to the reduced equation in the shock-layer subdomain may result in large errors, the reduced equation will be solved only in D_{OR}. The method will use the full equation in D_{IL} subject to data provided by the solution to the reduced

equation in the outer region. An analysis of the error induced with this procedure is presented in Corollary 2 below.

The local error bounds of Theorem 1 are now used to establish a global *a priori* error bound when using this procedure. The bound, as presented is sharp in D_{OR}; however, the bound reflects the crude error bound of Theorem 1 in the region of the shock.

COROLLARY 2. *Let u be the solution to* (1) *satisfying* (3-5). *Suppose v is obtained by first solving* (7) *in D_{OR} subject to* (3-5), *then solving* (1) *on D_{IL} with boundary data v on ∂D_{IL}. Assume that the boundary data* (3-5) *satisfy the compatibility conditions* (6),(8-9), *and that α, β and γ, with their first and second derivatives are all bounded. If $E = \|u - v\|_1$, then for ϵ small enough*

$$E = O(\epsilon) \tag{17}$$

in D_{OR}, and

$$E = O(\epsilon^{1/4} \ln^{1/2} \epsilon) \tag{18}$$

in D_{IL}.

In the proof, a simple bound on the size of the solution in the internal layer is established via a maximum principle argument. From there, the proof follows directly from Theorem 1.

Proof. The L_1 norm is defined as

$$\|g(x,t)\|_1 = \int_A |g(x,t)| dx dt.$$

Inequality (17) follows directly from applying the L_1 norm to the bound (13) of Theorem 1.

Let

$$\omega(t) = K_1 + K_2 t.$$

The constant K_2 may be chosen independent of ϵ such that

$$P[\omega] \geq P[v] = 0$$

for $(x,t) \in D_{IL}$, and the constant K_1 may be chosen independent of ϵ such that

$$\omega \geq v$$

for $(x,t) \in \partial D_{IL}$. Thus, the conditions of the statement of the maximum principle due to Nagumo and Westfal [18] are satisfied, and $\omega \geq v$ for all of D_{IL}. A symmetric argument can be used to establish a lower bounding function which has the same

form as ω. In addition, a similar argument can be used to establish upper and lower bounds for u in D_{IL}. Thus,

$$|u - v| = O(1)$$

for $(x,t) \in D_{IL}$. Since the area of D_{IL} is $O(\epsilon^{1/4} \ln^{1/2} \epsilon)$, applying the L_1 norm, the bound (18) follows. □

Thus far, the problem has been preconditioned by decomposing the domain into regions where the solution behaves differently, and forming subproblems in those regions. Another preconditioning for the problem in the internal-layer subdomain is also appropriate.

The preconditioning in the internal layer is a scaling and translation of the spatial coordinate. The translation allows the coordinate system to move with the shock by using $x - \Gamma(t)$ in place of x. The scaling is to stretch the spatial coordinate by $1/\epsilon$. This scaling is identified via multiple-scale asymptotic analysis [9,12,16], and allows the shock to be resolved in the local coordinate system. Combining these two, the new spatial coordinate in D_{IL} is

$$\tilde{x} = (x - \Gamma)/\epsilon. \tag{19}$$

The computational analog of this transformation is described in [15] or [10].

Asymptotics identified two subdomains and provided preconditioners for the problems within both subdomains. The domain decomposition algorithm described thus far requires *a priori* knowledge of the location of the shock. To remove this restriction, the domain decomposition is combined with a functional iteration. This allows for the iterative determination of the shock location in the computational method.

4. Iteration. The domain decomposition algorithm described in the previous section will now be treated as a single step in an iterative process. Each step of this iteration requires the solution of a linearized form of the reduced equation (7) in the outer-region subdomain followed by the solution of the full equation (1) in the internal-layer subdomain. Denote the iterate by $\hat{U}^{k+1}$. The equations governing the iterate are

$$\hat{U}_t^{k+1} + \hat{U}^k \hat{U}_x^{k+1} - r\hat{U}^{k+1} = 0 \tag{20}$$

in D_{OR}, and

$$\hat{U}_t^{k+1} + \hat{U}^{k+1} \hat{U}_x^{k+1} - \epsilon \hat{U}_{xx}^{k+1} - r\hat{U}^{k+1} = 0 \tag{21}$$

in D_{IL}. Boundary data for the internal-layer subdomain is provided by the solution of (20) in the outer-region subdomain.

In this section, the convergence of the iteration (20) in the outer-region subdomain to a solution of (7) will be established. In addition, a global *a priori* error bound for the method will be presented. Throughout this section the conditions on the boundary data presented in Section 2 are assumed to be satisfied.

THEOREM 3. *Let $\hat{U}^1, \hat{U}^2, \hat{U}^3, \ldots$ be the set of iterates of* (20) *in the subdomain D_{OR} satisfying the boundary data* (3-5) *with initial guess $\hat{U}^0$. Assume $\hat{U}^0$ satisfies* (3-5) *and is Lipschitz continuous on D. Let*

$$\delta = \sup_D |\hat{U}^k - \hat{U}^{k-1}|.$$

Then

$$|\hat{U}^{k+1} - \hat{U}^k| < \delta C e^{-\lambda t}(e^{Rt} - 1) \tag{22}$$

for $(x,t) \in D_{OR}$, where C, λ and R are known positive constants.

The proof utilizes some results on continuity of the iterates which will be established first. The boundedness of $\hat{U}^{k+1}$ is the subject of the following lemma.

LEMMA 4. *Let $\hat{U}^{k+1}$ be the solution to equation* (20) *on the subdomain D_{OR}, where $\hat{U}^k$ is Lipschitz continuous. Then,*

$$|\hat{U}^{k+1}| < \hat{K} e^{Rt},$$

for $(x,t) \in D_{OR}$, where $\hat{K}$ and R are constants independent of x, t and ϵ.

Proof. Consider the transformation $(x,t) \to (\xi,\tau)$ defined by

$$t = \tau, \tag{23}$$

and

$$\frac{\partial x^k}{\partial \tau} = \hat{U}^k(x^k(\xi,\tau),\tau), \tag{24}$$

with initial conditions

$$x^k(0) = \xi, \quad b > \xi > 0; \tag{25}$$

$$x^k(y_0^{-1}(\xi)) = 0, \quad \xi < 0; \tag{26}$$

$$x^k(y_b^{-1}(\xi)) = 0, \quad \xi > b. \tag{27}$$

Here, $(\xi,\tau) = (y_0(\tau),\tau)$ is the image of the curve $(x,t) = (0,t)$, and $(\xi,\tau) = (y_b(\tau),\tau)$ is the image of the curve $(x,t) = (b,t)$. Under this transformation, equation (20) becomes

$$\hat{U}_\tau^{k+1} = {}_\tau\hat{U}^{k+1}. \tag{28}$$

Since $\hat{U}^k$ is Lipschitz continuous, the transformation (23)-(27) is uniquely defined, and equation (28) may be used in place of equation (20).

The coefficient r is bounded, hence there is a constant $\hat{K}$ independent of x, t, and ϵ such that $\bar{\omega} = \hat{K}e^{Rr}$ is an upper bound for $\hat{U}^{k+1}$. In addition, $\underline{\omega} = -\bar{\omega}$ is a lower bound. Since $t = r$ the desired result is established. □

Based on the assumption of the boundedness of $\hat{U}_x^k$, the boundedness of $\hat{U}_x^{k+1}$ is established in Lemma 5.

LEMMA 5. *Suppose that the conditions of Lemma 4 obtain. Then $\hat{U}_x^{k+1}$ and $\hat{U}_t^{k+1}$ are bounded independent of x, t and ϵ in D_{OR}.*

In the proof of this lemma, an equation governing $\hat{U}_x^{k+1}$ will be derived. Then, a form of the maximum principle will be shown to apply after a change of the dependent variable.

Proof. Let $w = \hat{U}_x^{k+1}$. The equation

$$w_r = (r - \hat{U}_x^k)w + \frac{dr}{dx}\hat{U}^k, \tag{29}$$

is derived by taking the partial with respect to x of equation (20), then applying the transformation (23)-(27). Boundary data for w may be obtained by differentiating (3-5).

Let $w = e^{\lambda r}v$. The equation governing v is

$$v_r = (r - \hat{U}_x^k - \lambda)v + \frac{dr}{dx}\hat{U}^k.$$

Choose $\lambda = -\max(0, \inf_D(r(x) - \hat{U}_x^k))$, so that the coefficient of v in this equation will be nonnegative.

Define an upper bounding function as

$$\bar{\omega} = K_1\left(e^{K_2 r} - \frac{1}{2}\right),$$

where the constants K_1 and K_2 will be chosen. Then $z = \bar{\omega} - v$ satisfies

$$z_r = A(x,t)z + B(x,t), \tag{30}$$

where $A = r - \hat{U}_x^k - \lambda$ and $B = [K_2 - (r - \hat{U}_x^k - \lambda)]\bar{\omega} + \frac{K_1K_2}{2} - \frac{dr}{dx}\hat{U}^k$. The constants K_1 and K_2 may be chosen so that A, B and $z(x,0)$ are nonnegative. For example, choose

$$K_2 = \max(2, \sup_{D_{OR}}(r - \hat{U}_x^k - \lambda)) \text{ and } K_1 = \max(0, \sup_{D_{OR}}\frac{dr}{dx}\hat{U}^k, \sup_{\Pi} 2v).$$

Under these conditions, z is positive, and $\bar{\omega}$ is an upper bound for v.

A lower bounding function may be obtained in a similar way. Set $\underline{\omega} = K_3\left(e^{K_4 r} - \frac{1}{2}\right)$. If K_2 and λ are chosen as before and

$$K_3 = \min(0, \inf_{D_{OR}}\frac{dr}{dx}\hat{U}^k, \inf_{\Pi} 2v),$$

then $\underline{w} - v$ is nonpositive. Thus, $\underline{w}$ is a lower bounding function for v.

Since $\tau \leq T$, the boundedness of $\hat{U}_x^{k+1}$ independent of x, t and ϵ follows. The boundedness of $\hat{U}_t^{k+1}$ follows from the boundedness of the terms in equation (20). Therefore, the desired result is established. □

Given that $\hat{U}^0$ is Lipschitz continuous, Lemma 5 states that all of the remaining interates will be Lipschitz continuous. The significance of this is that the characteristic transformation (23)-(27) is uniquely defined. This is used in the proof of the theorem.

Proof. (Theorem 3). The equation governing $z = \hat{U}^k - \hat{U}^{k+1}$ in the characteristic coordinate system (23-27) is

$$z_\tau = (\hat{U}^{k-1} - \hat{U}^k)\hat{U}^{k+1} + rz.$$

An exponential change of variable will transform the problem such that a form of the maximum principle applies. Let

$$z = e^{-\lambda\tau}w,$$

where $\lambda = -\max(0, \inf r(x))$. Thus, the equation for w is

$$w_\tau = e^{\lambda t}(\hat{U}^{k-1} - \hat{U}^k)\hat{U}^{k+1} + \hat{r}w,$$

where $\hat{r} = r + \lambda$, and $w = 0$ on Π.

An upper bound for w may now be defined. Let

$$\omega(\tau) = \delta\frac{\eta}{R(e^{R\tau} - 1)},$$

where $R = \sup r - \inf r = \sup \hat{r}$. The constant η will be determined shortly. Taking the partial derivative with respect to τ,

$$\omega_\tau = \delta\eta + R\omega.$$

An equation for $f = \omega - w$ is

$$f_\tau = [\delta\eta - e^{\lambda\tau}(\hat{U}^{k-1} - \hat{U}^k)\hat{U}^{k+1} + (R - \hat{r})\omega] + \hat{r}f. \tag{31}$$

Since $\omega \geq 0$ and $w = 0$ on Π, the function f is nonnegative on Π. Choose $\eta = \max[0, \inf_D e^{\lambda t}\hat{U}^{k+1}]$. Thus, the source term in equation (31) is nonnegative, and $f \geq 0$ in D_{OR}. Thus $\omega(\tau) \geq w(\xi, \tau)$ for $(x(\xi,\tau), t(\xi,\tau))$ in D_{OR}. But $t = \tau$, hence

$$\delta K e^{-\lambda t}(e^{Rt} - 1) \geq z = \hat{U}^{k+1} - \hat{U}^k,$$

for $K = \eta/R$.

Using symmetric arguments, $-\omega$ can be shown to be a lower bound on z, leading to the desired result. □

This theorem provides an upper bound on the latest time for which the iteration converges. Apply the infinity norm to (22) to obtain

$$\|\hat{U}^{k+1} - \hat{U}^k\|_\infty \leq \hat{C}\|\hat{U}^k - \hat{U}^{k-1}\|_\infty.$$

Then the following corollary provides the conditions for convergence.

COROLLARY 6. *Suppose that the conditions of Theorem* 3 *obtain. Restrict the domain to* D_{OR}. *Let* $T_{\max}$ *be the largest positive number such that*

$$\hat{C} = \sup_{0 \leq t \leq T_{\max}} Ce^{-\lambda t}(e^{Rt} - 1) \leq 1.$$

Then the sequence of iterates defined by (20) *converges to a solution of* (7) *satisfying the data* (3-5) *for the upper bound* T *on time of equation* (2) *satisfying* $T_{\max} > T > 0$.

Proof. With the restriction of $t < T_{\max}$, the iteration is a contraction mapping, and the result follows. □

A statement of a global *a priori* error bound for the computational method is presented in Corollary 7 below. As with Corollary 2, the bound is sharp in D_{OR}; however, the bound is crude in the region of the shock.

COROLLARY 7. *Let* u *be the solution to* (1) *satisfying* (3-5). *Suppose each iterate* $\hat{U}^k$ *is obtained by first solving* (20) *in* D_{OR} *subject to* (3-5), *then solving* (21) *on* D_{IL} *with boundary data* $\hat{U}^k$ *on* ∂D_{IL}. *Suppose* $T_{\max} > T > 0$, *and let* $v = \lim_{k\to\infty} \hat{U}^k = \hat{U}^\infty$. *If* $E = \|u - v\|_1$, *then for* ϵ *small enough*

$$E = O(\epsilon) \tag{32}$$

in D_{OR}, *and*

$$E = O(\epsilon^{1/4} \ln^{1/2} \epsilon) \tag{33}$$

in D_{IL}. *Here* $\Delta = O(\epsilon^{1/4} \ln^{1/2} \epsilon)$.

To prove Corollary 7, it will first be established that $\hat{U}^\infty$ is the desired solution of equation (7). The bound in (32) will follow. Then, Lemma 8 will be applied to establish the bound (33).

LEMMA 8. *Let* v *be the solution to equation* (1) *on* D_{IL}. *Suppose that the data specified on* ∂D_{IL} *are bounded with bounded derivatives. Assume that* ∂D_{IL} *is at least* C^2. *Then*

$$|v| < K,$$

for $(x,t) \in D_{IL}$,

In the proof of this lemma, upper and lower bounding functions for v are established by applying the version of the maximum principle due to Nagumo and Westfal [18].

Proof. Make the change of dependent variable $w = e^{-\lambda t}v$, where λ satisfies $\hat{r} = r - \lambda \leq \hat{r}_0 < 0$. Then

$$P_\lambda[w] := w_t + e^{\lambda t} w w_x - \hat{r}w - \epsilon w_{xx} = 0.$$

where $w = e^{-\lambda t}v$ on ∂D_{IL}. Define

$$\bar{\omega} = \hat{K} = \max(0, \sup_{\partial D_{IL}} |e^{-\lambda t}v|).$$

Then

$$P_\lambda[\bar{\omega}] = -\hat{r}_0\hat{K} \geq P_\lambda[w].$$

In addition, $\bar{\omega} \geq e^{-\lambda t}v = w$ for $(x,t) \in \partial D_{IL}$. These conditions allow the application of the Nagumo-Westfal Lemma to conclude $\bar{\omega} \geq w$ for $(x,t) \in D_{IL}$. By symmetric arguments, $\underline{\omega} = -\bar{\omega}$ may be shown to be an lower bounding function on w. Setting $K = e^{\lambda T}\hat{K}$, the result is established. □

Proof (Corollary 7). In the outer region, $v = \hat{U}^\infty$, where bounds for $u - v = u - \hat{U}^\infty$ are given in (13-14). Thus, there is a constant K_{OR} such that $|u - \hat{U}^\infty| \leq \epsilon K_{OR}$ for $(x,t) \in D_{OR}$. Applying the L_1 norm to $u - \hat{U}^\infty$, relation (32) is established.

From Lemma 8, there is a constant K_{IL}, such that $|u| + |v| \leq K_{IL}$ in D_{IL}, where K_{IL} is independent of ϵ. Thus,

$$E \leq \int_{D_{IL}} K_{IL} ds.$$

Since the area covered by D_{IL} is of size $O(\epsilon^{1/4}\ln^{1/4}\epsilon)$, relation (33) holds. □

5. Concluding Remarks. A computational method will be created from the theory presented in the past few sections. The method is independent of the particular numerical schemes used; however, candidates for the numerical schemes will be discussed. The computational method is be constructed by solving equation (21) then solving equation (20) in succession. With minor modifications to account for the forcing term, the method of characteristics may be used for (21), or the equation my be solved using one of methods discussed in [1,17,3,14]. Equation (20) is solved in the local coordinate system (19). In this coordinate system, the coefficient of the diffusion is large enough to allow the application of standard finite difference methods.

In the most general case, the internal layer subdomain will change from iteration to iteration. The theory presented in this paper restricts consideration to a stationary internal-layer subdomain; however, experimental results demonstrate that this is not a constraint in the numerical method [10]. In addition, extensions to the theoretical basis to include a moving boundary are the subject of promising current research. When ∂D_{IL} is allowed to move between iterations, the method must be able to determine the location of the boundary as the computations proceed. This may be done by monitoring the Jacobian of the characteristic transformation (23-27), as discussed

in [10]. In addition, Lemma 5 suggests that monitoring $\hat{U}_x^k$ may be used to place the boundary. The value of $\hat{U}_x^k$ will be large and negative in a neighborhood of the shock, and it will be bounded independent of ϵ in the outer-region.

This paper demonstrated the use of asymptotics to dictate a numerical method with high accuracy and efficiency. Asymptotic analysis provided a theoretical basis for a domain decomposition, and guided in the derivation of rigorous local and global *a priori* error bounds.

The asymptotic analysis can be used to analyze existing methods as well as develop new algorithms. For example, the analysis may be used to study the effects of using artificial viscosity. Theorem 1 provides an *a priori* upper bound on the error induced by using $\hat{\epsilon}$ in place of ϵ where $\hat{\epsilon} > \epsilon$. Such an equation would be solved when using constant coefficient artificial diffusion. This bound may be obtained by using the theorem to establish bounds separately on $|u(x,t,\hat{\epsilon}) - U|$ and $|u(x,t,\epsilon) - U|$, then summing them.

The availability of estimates and bounds on the error is important in the design of numerical methods. Rigorous *a priori* error bounds were established for the method presented here. In addition, the particular numerical schemes used for the subproblems allowed *a posteriori* error estimation. The *a priori* error bounds were shown to be much larger than the errors observed in the computations; thus, sharper error bounds are expected.

Acknowledgements. The author would like to akcnowledge the financial and moral support of Ahmed Sameh during the research of this project. Dan Sorensen's ability to identify significant ideas, and suppress insignificant ones was invaluable. In addition, the development of this method would not have been possible without Gerald Hedstrom, Raymond C. Chin, and Fred Howe's work in the area and collaboration on this project.

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NASA
National Aeronautics and Space Administration

Report Documentation Page

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR-181734 ICASE Report No. 88-62		

4. Title and Subtitle	5. Report Date
A PHYSICALLY MOTIVATED DOMAIN DECOMPOSITION FOR SINGULARLY PERTURBED EQUATIONS	October 1988
	6. Performing Organization Code

7. Author(s)	8. Performing Organization Report No.
Jeffrey S. Scroggs	88-62
	10. Work Unit No.
	505-90-21-01

9. Performing Organization Name and Address	11. Contract or Grant No.
Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225	NAS1-18605
	13. Type of Report and Period Covered
	Contractor Report

12. Sponsoring Agency Name and Address	14. Sponsoring Agency Code
National Aeronautics and Space Administration Langley Research Center Hampton, VA 23665-5225	

15. Supplementary Notes

Langley Technical Monitor:
Richard W. Barnwell

Submitted to the SIAM
Journal of Numerical Analysis

Final Report

16. Abstract

A domain decomposition algorithm suitable for the efficient and accurate solution of a parabolic reaction convection diffusion equation with small parameter on the diffusion term is presented. Convergence is established via maximum principle arguments. The equation arises in the modeling of laminar transonic flow. Decomposition into subdomains is accomplished via singular perturbation analysis which dictates regions where certain reduced equations may be solved in place of the full equation, effectively preconditioning the problem. This paper concentrates on the theoretical basis of the method, establishing local and global a priori error bounds.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
algorithm, computational fluid dynamics, asymptotics	34 - Fluid Mechanics and Heat Transfer 64 - Numerical Analysis Unclassified - unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of pages	22. Price
Unclassified	Unclassified	15	A02